A characterization of Schur multipliers between character-automorphic Hardy spaces

Daniel Alpay, Mamadou Mboup

A characterization of Schur multipliers between character-automorphic Hardy spaces

D. Alpay and M. Mboup

Abstract

We give a new characterization of character-automorphic Hardy spaces of order 2 and of their contractive multipliers in terms of de Branges Rovnyak spaces. Keys tools in our arguments are analytic extension and a factorization result for matrix-valued analytic functions due to Leech.

Keywords. Character-automorphiuc functions, Hardy spaces, de Branges Rovnyak spaces, Schur multipliers.

Mathematics Subject Classification (2000). Primary: 30F35, 46E22. Secondary: 30B40

1 Introduction

Let Γ be a Fuchsian group of Möbius transformations of the unit disk $\mathbb{D} = \{z \in \mathbb{C}\,;\, |z| < 1\}$ onto itself. For $1 \leqslant p \leqslant \infty$ and for any character α of Γ, we consider the spaces

$$\mathcal{H}_p^\alpha = \{f \in \mathcal{H}_p \ \mid \ f \circ \gamma = \alpha(\gamma) f, \quad \forall\, \gamma \in \Gamma\}\,.$$

These spaces are called character-automorphic Hardy spaces. A characterization of such spaces in terms of Poincaré theta series may be found in [15], [18], [13], [9]. In particular, Pommerenke showed in [15] that the series

$$f(z) = \frac{b_0(z)}{b_0'(z)} \sum_{\gamma\in\Gamma} \overline{\alpha(\gamma)}\theta(\gamma(z))h(\gamma(z))\frac{\gamma'(z)}{\gamma(z)} \tag{1.1}$$

defines a bounded linear operator from the classical Hardy space $\mathcal{H}_p(\mathbb{D})$ into the subspace $\mathcal{H}_p^\alpha(\mathbb{D})$.

In the present paper we restrict ourselves to the case $p = 2$. We first give a characterization of the character-automorphic Hardy space $\mathcal{H}_2^\alpha(\mathbb{D})$ in terms of an associated de Branges Rovnyak space of functions analytic in the open unit disk; see Theorem 3.2. We also characterize the contractive multipliers between $\mathcal{H}_2^{\overline{\beta}\alpha}(\mathbb{D})$ and $\mathcal{H}_2^\alpha(\mathbb{D})$, where α and β two given characters; see Theorem 4.2. Our method is mainly based on analytic extension of positive kernels and factorization results from Nevanlinna-Pick interpolation theory.

2 A review on character-automorphic Hardy spaces

2.1 Fuchsian groups and automorphic functions

Let G be a group of linear transformations, $T(z) = \frac{az+b}{cz+d}$, $ad - bc = 1$, in the complex plane and let ι denotes the identity transformation. Two points z and z' in $\mathbb{C}$ are said to be *congruent* with respect to G, if $z' = T(z)$ for some $T \in G$ and $T \neq \iota$. Two regions $R, R' \subset \mathbb{C}$ are said to be G-congruent or G-equivalent if there exists a transformation $T \neq \iota$ which sends R to R'. A region R which does not contain any two G-congruent points and such that the neighborhood of any point on the boundary contains G-congruent points of R is called a *fundamental region* for G. A *properly discontinuous* group is a group G having a fundamental region [10]. This amounts to saying that the identity transformation is isolated.

Definition 2.1 *A Fuchsian group is a properly discontinuous group each of whose transformation maps $\mathbb{D}$, $\mathbb{T}$ and $\mathbb{C}\backslash\overline{\mathbb{D}}$ onto themselves.*

A Fuchsian group Γ is said to be of convergence type (see *e.g.* [15]) if

$$\sum_{\gamma\in\Gamma} \left(1 - |\gamma(z)|^2\right) = \left(1 - |z|^2\right) \sum_{\gamma\in\Gamma} |\gamma'(z)| < \infty \quad z \in \mathbb{D}.$$

Then, the Green's function [15] of Γ with respect to a point $\xi \in \mathbb{D}$ is defined as the Blaschke product

$$b_\xi(z) = \prod_{\gamma\in\Gamma} \frac{\gamma(\xi) - z}{1 - \overline{\gamma(\xi)}z} \frac{|\gamma(\xi)|}{\gamma(\xi)}. \tag{2.2}$$

It satisfies

$$b_\xi(\varphi(z)) = \mu_\xi(\varphi) b_\xi(z), \quad \forall \varphi \in \Gamma, \tag{2.3}$$

where μ_ξ is the character of Γ associated with $b_\xi(z)$. A function satisfying the relation (2.3) is said to be *character-automorphic* with respect to Γ while a Γ-periodic function, as for example $|b_\xi(z)| = |b_\xi(\varphi(z))|$, is called *automorphic* with respect to Γ.

2.2 Spaces of character-automorphic functions

We now briefly mention the main properties pertaining to spaces of character-automorphic functions. The materials presented here are essentially borrowed from [15] and [19] (see also [11, 18]). Let $\widehat{\Gamma}$ be the dual group of Γ, *i.e.* the group of (unimodular) characters. For an arbitrary character $\alpha \in \widehat{\Gamma}$, associate the subspaces of the classical space $L_2(\mathbb{T})$

$$L_2^\alpha = \{f \in L_2 \,|\, f \circ \gamma = \alpha(\gamma) f, \, \forall \gamma \in \Gamma\}$$
$$\mathcal{H}_2^\alpha(\mathbb{D}) = L_2^\alpha \bigcap \mathcal{H}_2(\mathbb{D})$$

Let Γ be a Fuchsian group without elliptic and parabolic element. We say that Γ is of Widom type if, and only if, the derivative of $b_0(z)$ is of bounded characteristic. In this case, Widom [20] has shown that the space $\mathcal{H}_\infty^\alpha$ is not trivial for any character $\alpha \in \widehat{\Gamma}$ and we have

Theorem 2.2 (Pommerenke [15]) *Let Γ be of Widom type and let $\theta(z)$ be the inner factor of $b_0'(z)$. If α is any character of Γ and if $h(z)$ is in $\mathcal{H}_p(\mathbb{D}), 1 \leqslant p \leqslant \infty$, then the function defined by* (1.1) *is in $\mathcal{H}_p^\alpha(\mathbb{D})$ and*

$$\|f\|_p \leqslant \|h\|_p, \quad f(0) = \theta(0) h(0).$$

The Poincaré series [14] in (1.1) thus defines, in particular, a projection: $P^\alpha : \theta\mathcal{H}_2(\mathbb{D}) \to \mathcal{H}_2^\alpha(\mathbb{D})$. An important property of the space $\mathcal{H}_2^\alpha(\mathbb{D})$ that we will need is that, point evaluation $f \mapsto f(\xi), \xi \in \mathbb{D}$ is a bounded linear functional. The space therefore admits a reproducing kernel k^α:

$$\langle f(z), k^\alpha(z,\xi)\rangle_{\mathcal{H}_2^\alpha(\mathbb{D})} = f(\xi), \, \xi \in \mathbb{D} \quad \text{for all } f \in \mathcal{H}_2^\alpha(\mathbb{D})$$
$$\text{with } k^\alpha(z,\xi) \in \mathcal{H}_2^\alpha(\mathbb{D}), \quad \forall\, \xi \in \mathbb{D}$$

Since $\mathcal{H}_2^\alpha(\mathbb{D}) \neq \{const\}$, we have $k^\alpha(\xi,\xi) = \|k^\alpha(\cdot,\xi)\|^2_{\mathcal{H}_2^\alpha(\mathbb{D})} > 0$ for every $\xi \in \mathbb{D}$. In the sequel, the Green's function $b_0(z)$ with respect to 0, will be denoted by $b(z)$ for short.

Let Γ be a group of Widom type and let E be associated[1] to it in such a way that $\overline{\mathbb{C}}\backslash E$ be equivalent to the Riemann surface $\mathbb{D}/\Gamma$, obtained by identifying Γ-congruent points. Then, there exists a universal covering map $\mathfrak{z} : \mathbb{D} \to \overline{\mathbb{C}}\backslash E \simeq \mathbb{D}/\Gamma$ such that

- $\mathfrak{z}$ maps $\mathbb{D}$ conformally onto $\overline{\mathbb{C}}\backslash E$,
- $\mathfrak{z}$ is automorphic with respect to Γ: $\mathfrak{z} \circ \gamma = \mathfrak{z}, \quad \forall\, \gamma \in \Gamma$
- and $\mathfrak{z}(z_1) = \mathfrak{z}(z_2) \Rightarrow \exists \gamma \in \Gamma \mid z_1 = \gamma(z_2)$

In particular, $\mathfrak{z}$ maps one-to-one the *normal fundamental domain* of Γ with respect to the origin,

$$\mathcal{F} = \{z \in \mathbb{D} \,:\, |\gamma'(z)| < 1 \quad \text{for all } \gamma \in \Gamma, \gamma \neq \iota\} \tag{2.4}$$

conformally onto some sub-domain of $\overline{\mathbb{C}}\backslash E$. We assume that $\mathfrak{z}$ is normalized so that $(\mathfrak{z}b)(0)$ is real and positive. In all the sequel, the character associated to the Green's function $b(z)$ will be denoted by μ. The starting point of the next section is the following result:

Lemma 2.3 ([12]) *The reproducing kernel for the space $\mathcal{H}_2^\alpha(\mathbb{D})$ has the form*

$$k^\alpha(z,\omega) = c(\alpha)\frac{\frac{k^{\alpha\mu}(z,0)}{b(z)}k^\alpha(\omega,0)^* - \left(\frac{k^{\alpha\mu}(\omega,0)}{b(\omega)}\right)^* k^\alpha(z,0)}{\mathfrak{z}(z) - \mathfrak{z}(\omega)^*} \tag{2.5}$$

where

$$c(\alpha) = \frac{\mathfrak{z}(0)b(0)}{k^{\alpha\mu}(0,0)} > 0. \tag{2.6}$$

3 An associated de Branges-Rovnyak space

In this section we give a characterization of the space $\mathcal{H}_2^\alpha(\mathbb{D})$ in terms of an associated de Branges Rovnyak space. To begin, let

$$\Omega_+ = \{z \in \mathbb{D}\,;\, \mathrm{Im}\ \mathfrak{z}(z) > 0\}\,.$$

[1] See [21] for an example of a construction of a group Γ associated with a finite union of disjoint arcs of the unit circle.

Setting

$$A^{\alpha}(z) = \sqrt{\frac{c(\alpha)}{2}}\left(\frac{k^{\alpha\mu}(z,0)}{b(z)} + ik^{\alpha}(z,0)\right)$$
$$B^{\alpha}(z) = \sqrt{\frac{c(\alpha)}{2}}\left(\frac{k^{\alpha\mu}(z,0)}{b(z)} - ik^{\alpha}(z,0)\right)$$

we can rewrite the reproducing kernel k^{α} as

$$k^{\alpha}(z,\omega) = \frac{2A^{\alpha}(z)}{1 - i\mathfrak{z}(z)}\frac{1 - S_{\alpha}(z)S_{\alpha}(w)^*}{1 - \sigma(z)\sigma(w)^*}\frac{2A^{\alpha}(w)^*}{1 + i\mathfrak{z}(w)^*} \tag{3.1}$$

where $S_{\alpha}(z) = B^{\alpha}(z)/A^{\alpha}(z)$ and $\sigma(z) = \frac{1+i\mathfrak{z}(z)}{1-i\mathfrak{z}(z)}$. We note that the functions $A^{\alpha}(z)$ and $B^{\alpha}(z)$ are character-automorphic with the same character α while S_{α} and σ are automorphic functions. From now on, the notation $f^{\nu}(z)$ will means that the function $f^{\nu}(z)$ is character-automorphic with the superscript $\nu \in \widehat{\Gamma}$ being the associated character, and the notation $f_{\nu}(z)$ will stand for a function depending on the character ν (automorphic or not).

Proposition 3.1 *There exists a Schur function $\mathscr{S}_{\alpha}$ such that $S_{\alpha}(z) = \mathscr{S}_{\alpha}(\sigma(z))$.*

Proof: Since the kernel $k^{\alpha}(z,\omega)$ is positive in $\mathbb{D}$, and hence in Ω_+, it is clear that $\frac{1-S_{\alpha}(z)S_{\alpha}(w)^*}{1-\sigma(z)\sigma(w)^*}$ is also positive in Ω_+. Now, observe that the function σ maps $\Omega_+ \cap \mathcal{F}$ into some subset $\Delta \subset \mathbb{D}$ and this mapping is one-to-one. Let ς be given by: $(\varsigma \circ \sigma)(z) = z$, $\forall z \in \Omega_+ \cap \mathcal{F}$ (in particular this will also hold for any region congruent to $\Omega_+ \cap \mathcal{F}$) and let the function $\widetilde{\mathscr{S}_{\alpha}}$ be defined on Δ by:

$$\widetilde{\mathscr{S}_{\alpha}}(\lambda) = (S_{\alpha} \circ \varsigma)(\lambda),\ \forall \lambda \in \Delta.$$

Then it comes that the kernel

$$\frac{1 - \widetilde{\mathscr{S}_{\alpha}}(\lambda)\widetilde{\mathscr{S}_{\alpha}}(\mu)^*}{1 - \lambda\mu^*}$$

is positive on Δ. Now, this implies (see for instance [1, Theorem 2.6.5]) the existence of a unique extension of $\widetilde{\mathscr{S}_{\alpha}}(\lambda)$, analytic and contractive in all $\mathbb{D}$. We subsequently call $\mathscr{S}_{\alpha}(\lambda)$ this extension, and denote by $\mathcal{H}(\mathscr{S}_{\alpha})$ the reproducing kernel Hilbert space with reproducing kernel

$$K_{\mathscr{S}_{\alpha}}(\lambda,\mu) = \frac{1 - \mathscr{S}_{\alpha}(\lambda)\mathscr{S}_{\alpha}(\mu)^*}{1 - \lambda\mu^*}$$

By construction, the equality

$$S_\alpha(z) = \mathscr{S}_\alpha(\sigma(z))$$

holds for all $z \in \Omega_+ \cap \mathcal{F}$. Since $S_\alpha(z)$ is analytic in $\mathbb{D}$, it must also hold for all $\mathbb{D}$. $\square$

In connection with the previous proposition and the next theorem, we recall that reproducing kernel Hilbert spaces $\mathcal{H}(\mathscr{S})$ of functions analytic in the open unit disk and with a reproducing kernel of the form

$$\frac{1 - \mathscr{S}(\lambda)\mathscr{S}(\mu)^*}{1 - \lambda\mu^*}$$

were introduced and studied by de Branges and Rovnyak; see [5, Appendix], [6]. We also refer the reader to [8] and [1] for more information on these and on related spaces.

Theorem 3.2 *The character-automorphic Hardy space $\mathcal{H}_2^\alpha(\mathbb{D})$ can be described as*

$$\mathcal{H}_2^\alpha(\mathbb{D}) = \left\{ F(z) = \frac{\sqrt{2}A^\alpha(z)}{1 - i\mathfrak{z}(z)} f(\sigma(z)) \; ; f \in \mathcal{H}(\mathscr{S}_\alpha) \right\} \tag{3.2}$$

with the norm

$$\|F\|_{\mathcal{H}_2^\alpha(\mathbb{D})} = \|f\|_{\mathcal{H}(\mathscr{S}_\alpha)}.$$

Proof: Recall that the map which to $F \in \mathcal{H}_2^\alpha(\mathbb{D})$ associates its restriction $F|_{\Omega_+}$ to Ω_+ is an isometry from $\mathcal{H}_2^\alpha(\mathbb{D})$ onto the reproducing kernel Hilbert space with reproducing $k^\alpha(z, \omega)$ defined by (2.5), where z, ω are now restricted to Ω_+. We denote this last space by $\mathcal{H}_2^\alpha(\mathbb{D})\big|_{\Omega_+}$. By Proposition 3.1 and using (3.1) we see that the operator of multiplication by $\frac{2A^\alpha(z)}{1-i\mathfrak{z}(z)}$ is an isometry from the reproducing kernel Hilbert space $\mathcal{H}$ with reproducing kernel

$$\frac{1 - \mathscr{S}_\alpha(\sigma(z))\mathscr{S}_\alpha(\sigma(w)^*}{1 - \sigma(z)\sigma(w)^*}$$

onto $\mathcal{H}_2^\alpha(\mathbb{D})\big|_{\Omega_+}$. Furthermore, the composition map by σ is an isometry from the de Branges Rovnyak space $\mathcal{H}(\mathscr{S}_\alpha)$ onto $\mathcal{H}$. We have that

$$\mathcal{H} = \{ f \circ \sigma \; ; \; f \in \mathcal{H}(\mathscr{S}_\alpha) \},$$

with norm $\|f \circ \sigma\|_{\mathcal{H}} = \|f\|_{\mathcal{H}(\mathscr{S}_\alpha)}$, as follows from the equalities

$$f(\sigma(\omega)) = \langle f(\cdot), K_{\mathscr{S}_\alpha}(\cdot, \sigma(\omega))\rangle_{\mathcal{H}(\mathscr{S}_\alpha)} = \langle f \circ \sigma(\cdot), K_{\mathscr{S}_\alpha}(\sigma(\cdot), \sigma(\omega))\rangle_{\mathcal{H}}$$

Thus the restrictions of the elements of $\mathcal{H}_2^\alpha(\mathbb{D})$ to Ω_+ are of the form as in (3.2) for z restricted to Ω_+. By analytic extension, the elements of $\mathcal{H}_2^\alpha(\mathbb{D})$ have the same form in the whole of $\mathbb{D}$. $\square$

4 Schur multipliers

A Schur function is a function analytic and contractive in the open unit disk. Equivalently, it is a function s such that the operator of multiplication by s is a contraction from the classical Hardy of the open unit disk into itself. This last definition is our starting point to define character-automorphic Schur multipliers.

Definition 4.1 *A character-automorphic function $s^\beta(z)$, with character β, will be called a Schur multiplier if the operator of multiplication by $s^\beta(z)$ is a contraction from $\mathcal{H}_2^{\overline{\beta}\alpha}(\mathbb{D})$ into $\mathcal{H}_2^\alpha(\mathbb{D})$.*

Equivalently, the character-automorphic function $s^\beta(z)$ is a Schur multiplier if and only if the kernel

$$K_{s^\beta}^\alpha(z, w) = k^\alpha(z, w) - s^\beta(z)s^\beta(w)^* k^{\overline{\beta}\alpha}(z, w) \tag{4.1}$$

is positive in $\mathbb{D}$. The kernel $K_{s^\beta}(z, w)$ is in particular positive in Ω_+, and we will consider it in Ω_+. We note that in view of [7, Lemma 2, p. 142], [4, Theorem 1.1.4, p. 10], the positivity of the analytic kernel K_{s^β} on Ω_+ implies its positivity on $\mathbb{D}$.

Theorem 4.2 *A character-automorphic function s^β is a Schur multiplier if and only if there exists a $\mathbb{C}^{2\times 2}$–matrix valued Schur function $\Sigma(z)$ such that*

$$s^\beta(z) = \frac{A^\alpha(z)}{A^{\overline{\beta}\alpha}(z)} \frac{\Sigma_{12}(\sigma(z))}{1 - S_{\overline{\beta}\alpha}(z)\Sigma_{22}(\sigma(z))} \tag{4.2}$$

$$S_\alpha(z) = \Sigma_{11}(\sigma(z)) + \frac{\Sigma_{12}(\sigma(z))S_{\overline{\beta}\alpha}(z)\Sigma_{21}(\sigma(z))}{1 - S_{\overline{\beta}\alpha}(z)\Sigma_{22}(\sigma(z))} \tag{4.3}$$

Proof: The positivity of the kernel (4.1) in Ω_+ is equivalent to the positivity in Ω_+ of the kernel

$$K_{\mathscr{S}_\alpha}(\sigma(z),\sigma(\omega)) - T(z)T(\omega)^* K_{\mathscr{S}_{\overline{\beta}\alpha}}(\sigma(z),\sigma(\omega)),$$

where

$$T(z) = \frac{A^{\overline{\beta}\alpha}(z)}{A^\alpha(z)} s^\beta(z).$$

As in the proof of Proposition 3.1, we note that the function σ maps $\Omega_+ \cap \mathcal{F}$ into some subset $\Delta \subset \mathbb{D}$ and this mapping is one-to-one, and consider again the function ς defined by: $(\varsigma \circ \sigma)(z) = z$, $\forall z \in \Omega_+ \cap \mathcal{F}$. The kernel

$$K_{\mathscr{S}_\alpha}(\lambda,\mu) - T(\varsigma(\lambda))T(\varsigma(\mu))^* K_{\mathscr{S}_{\overline{\beta}\alpha}}(\lambda,\mu), \tag{4.4}$$

is positive in Δ. We now show that $T \circ \varsigma$ admits an analytic extension to $\mathbb{D}$. To that purpose we consider the linear relation in $\mathcal{H}(\mathscr{S}_\alpha) \times \mathcal{H}(\mathscr{S}_{\overline{\beta}\alpha})$ spanned by the elements of the form

$$(K_{\mathscr{S}_\alpha}(\cdot,\omega), T(\varsigma(\omega))^* K_{\mathscr{S}_{\overline{\beta}\alpha}}(\cdot,\omega)), \quad \omega \in \Delta.$$

It is densely defined and it is contractive because of the positivity of the kernel (4.4) in Δ. It is therefore the graph of a densely defined contraction $\widetilde{X}$. We note its extension to $\mathcal{H}(\mathscr{S}_\alpha)$ by X. For $\omega \in \Delta$ and $f \in \mathcal{H}(\mathscr{S}_\alpha)$ we have

$$\begin{aligned}(X^*f)(\omega) &= \langle X^*f, K_{\mathscr{S}_\alpha}(\cdot,\omega)\rangle_{\mathcal{H}(\mathscr{S}_\alpha)} \\ &= \langle f, T(\varsigma(\omega))^* K_{\mathscr{S}_{\overline{\beta}\alpha}}(\cdot,\omega)\rangle_{\mathcal{H}(\mathscr{S}_\alpha)} \\ &= T(\varsigma(\omega))f(\omega).\end{aligned}$$

Let $f_0(\lambda) = K_{\mathscr{S}_{\overline{\beta}\alpha}}(\lambda,\omega_0)$ where $\omega_0 \in \mathbb{D}$. We have

$$T(\varsigma(\lambda)) = \frac{(X^*f_0)(\lambda)}{f_0(\lambda)}, \quad \lambda \in \Delta.$$

It follows that $T \circ \varsigma$ has an analytic extension to $\mathbb{D}$, which we will denote by $\mathscr{R}$. Thus the kernel

$$K_{\mathscr{S}_\alpha}(\lambda,\mu) - \mathscr{R}(\lambda)\mathscr{R}(\mu)^* K_{\mathscr{S}_{\overline{\beta}\alpha}}(\lambda,\mu), \tag{4.5}$$

is analytic in λ and μ^* in $\mathbb{D}$. Therefore it is still positive in $\mathbb{D}$; see [7, Lemma 2, p. 142], [4, Theorem 1.1.4, p. 10]. By [2, Theorem 11.1, p. 61], a necessary and sufficient condition for the kernel (4.5) to be positive is that there exists a $\mathbb{C}^{2\times 2}$–matrix valued Schur function $\Sigma(\lambda)$ such that

$$\mathscr{R}(\lambda) = \frac{\Sigma_{12}(\lambda)}{1-\mathscr{S}_{\overline{\beta}\alpha}(\lambda)\Sigma_{22}(\lambda)} \tag{4.6}$$

$$\mathscr{S}_{\alpha}(\lambda) = \Sigma_{11}(\lambda) + \frac{\Sigma_{12}(\lambda)\mathscr{S}_{\overline{\beta}\alpha}(\lambda)\Sigma_{21}(\lambda)}{1-\mathscr{S}_{\overline{\beta}\alpha}(\lambda)\Sigma_{22}(\lambda)}. \tag{4.7}$$

The above mentioned result from [2] stems from rewriting the positive kernel (4.5) as

$$\frac{\mathcal{A}(\lambda)\mathcal{A}(\mu)^* - \mathcal{B}(\lambda)\mathcal{B}(\mu)^*}{1-\lambda\mu^*}$$

where

$$\begin{aligned}\mathcal{A}(\lambda) &= \begin{pmatrix}1 & \mathscr{R}(\lambda)\mathscr{S}_{\overline{\beta}\alpha}(\lambda)\end{pmatrix}\\ \mathcal{B}(\lambda) &= \begin{pmatrix}\mathscr{S}_{\alpha}(\lambda) & \mathscr{R}(\lambda)\end{pmatrix},\end{aligned}$$

and using a factorization result known as Leech's theorem, which insures the existence of a $\mathbb{C}^{2\times 2}$-valued Schur function Σ such that

$$\mathcal{B}(z) = \mathcal{A}(z)\Sigma(z),$$

from which (4.6) follows.

This unpublished result of R.B. Leech has been proved using the commutant lifting theorem by M. Rosenblum; see [16, Theorem 2, p. 134] and [17, Example 1, p. 107]. Further discussions and applications can also be found in [3]. It can also be proved using tangential Nevanlinna-Pick interpolation and Montel's theorem.

Finally we replace in (4.6) λ by $\sigma(z)$ where $z \in \Omega_+ \cap \mathcal{F}$. We obtain the formulas in the statement of the theorem for $z \in \Omega_+ \cap \mathcal{F}$, and hence for $z \in \mathbb{D}$ by analytic extension. □

Acknowledgments. The first author thanks the Earl Katz family for endowing the chair which supports his research. This work was done while the second author was visiting the department of mathematics of Ben-Gurion University. The second author thanks the Center for Advanced Studies in Mathematics (CASM) of the department of mathematics of Ben-Gurion University which supported his stay.

References

[1] D. Alpay. *Algorithme de Schur, espaces à noyau reproduisant et théorie des systèmes*, volume 6 of *Panoramas et Synthèses*. Société Mathématique de France, Paris, 1998.

[2] D. Alpay and V. Bolotnikov. On tangential interpolation in reproducing kernel Hilbert space modules and applications. In H. Dym, B. Fritzsche, V. Katsnelson, and B. Kirstein, editors, *Topics in interpolation theory*, volume 95 of *Operator Theory: Advances and Applications*, pages 37–68. Birkhäuser Verlag, Basel, 1997.

[3] D. Alpay, P. Dewilde, and H. Dym. On the existence and construction of solutions to the partial lossless inverse scattering problem with applications to estimation theory. *IEEE Trans. Inform. Theory*, 35:1184–1205, 1989.

[4] D. Alpay, A. Dijksma, J. Rovnyak, and H. de Snoo. *Schur functions, operator colligations, and reproducing kernel Pontryagin spaces*, volume 96 of *Operator theory: Advances and Applications*. Birkhäuser Verlag, Basel, 1997.

[5] L. de Branges and J. Rovnyak. Canonical models in quantum scattering theory. In C. Wilcox, editor, *Perturbation theory and its applications in quantum mechanics*, pages 295–392. Wiley, New York, 1966.

[6] L. de Branges and J. Rovnyak. *Square summable power series*. Holt, Rinehart and Winston, New York, 1966.

[7] W.F. Donoghue. *Monotone matrix functions and analytic continuation*, volume 207 of *Die Grundlehren der mathematischen Wissennschaften*. Springer–Verlag, 1974.

[8] H. Dym. *J–contractive matrix functions, reproducing kernel Hilbert spaces and interpolation*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1989.

[9] C. J. Earle and A. Marden. On Poincaré series with application to H^p spaces on bordered Riemann surfaces. *Illinois J. Math.*, 13:202–219, 1969.

[10] L.R. Ford. *Automorphic functions*. Chelsea, 2nd edition, 1951.

[11] M. Hasumi. *Hardy classes on infinitely connected Riemann surfaces*, volume 1027 of *Lecture notes in mathematics*. Springer–Verlag, 1983.

[12] S. Kupin and P. Yuditskii. Analogs of the Nehari and Sarason theorems for character-automorphic functions and some related questions. In *Topics in interpolation theory (Leipzig, 1994)*, volume 95 of *Oper. Theory Adv. Appl.*, pages 373–390. Birkhäuser, Basel, 1997.

[13] T. A. Metzger and K. V. Rajeswara Rao. Approximation of Fuchsian groups and automorphic forms of dimension -2. *Indiana Univ. Math. J.*, 21:937–949, 1971/72.

[14] H. Poincaré. Sur l'uniformisation des fonctions analytiques. *Acta Math.*, 31(1):1–63, 1908.

[15] Ch. Pommerenke. On the Green's function of Fuchsian groups. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 2:409–427, 1976.

[16] M. Rosenblum. A corona theorem for countably many functions. *Integral Equations and Operator Theory*, 3:125–137, 1980.

[17] M. Rosenblum and J. Rovnyak. *Hardy classes and operator theory*. Birkhäuser Verlag, Basel, 1985.

[18] M. V. Samokhin. Some classical problems in the theory of analytic functions in domains of Parreau-Widom type. *Mat. Sb.*, 182(6):892–910, 1991.

[19] M. Sodin and P. Yuditskii. Almost periodic Jacobi matrices with homogeneous spectrum, infinite-dimensional Jacobi inversion, and Hardy spaces of character-automorphic functions. *J. Geom. Anal.*, 7(3):387–435, 1997.

[20] H. Widom. $\mathcal{H}_p$ sections of vector bundles over Riemann surfaces. *Ann. of Math. (2)*, 94:304–324, 1971.

[21] P. Yuditskii. Two remarks on Fuchsian groups of Widom type. In *Operator theory, system theory and related topics (Beer-Sheva/Rehovot, 1997)*, volume 123 of *Oper. Theory Adv. Appl.*, pages 527–537. Birkhäuser, Basel, 2001.

Daniel Alpay
Department of mathematics
Ben-Gurion University of the Negev
P.O.B. 653 Beer-Sheva 84105 - Israel
E-mail: dany@cs.bgu.ac.il

Mamadou Mboup
UFR mathématiques et informatique & EPI ALIEN INRIA
Université Paris Descartes
45, rue des Saints-Pères, 75006 Paris, France
E-mail: mboup@math-info.univ-paris5.fr